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Temperature fields in a system consisting of two semi-infinite rods (Part II)

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General expressions are presented for temperature fields in a system consisting of two semi-infinite rods, when the lateral surfaces of the rods are thermally insulated. Results have been discussed in particular cases, namely, when one of the rods is heated by (*i*) a continuous heat source, (*ii*) a discrete moving heat source.

Symbols Used

a_1, a_2, thermal diffusivity coefficients,

a, characteristic thermal diffusivity,

$M_1 M_2 = a^{-1}[a_1, a_2]$, dimensionless thermal diffusivity,

r space coordinate,

ρ, characteristic length,

$x = \frac{r}{\rho}$, dimensionless space coordinate,

t, time variable,

$F_0 = \frac{at}{\rho^2}$, Fourier number

$T_1(r, t)$, temperature distribution in first rod,

$T_2(r, t)$, temperature distribution in second rod,

T_0, characteristic temperature,

$\theta_1(x, F_0), \theta_2(x, F_0) = T_0^{-1}[T_1(r, t), T_2(r, t)]$, dimensionless temperature distributions in first and second rod, respectively,

$W(r, t)$, volume heat source,

c, specific heat capacity,

γ, density,

$P_0(x, F_0) = W(r, t) \frac{\rho^2}{ac YT_0}$, Pomerantensev criterion,

$K_\lambda = \frac{\lambda_1}{\lambda_2}$, the parameter characterizing the relative conductivity,

$K_a = \frac{a_1}{a_2}$, the parameter characterizing the thermal inertia of one body relative to that of other,

$K_s = \frac{K_\lambda}{(K_a)^{\frac{1}{2}}}$, the parameter characterizing the thermal activity of the first rod relative to that of the second,

$i' = \sqrt{-1}$,

$i^n \operatorname{erfc} x = \int_x^\infty i^{n-1} \operatorname{erfc} u\,du$,

δ, Dirac delta function.

Introduction

Several years back Smirnov (1958) studied a heat conduction problem for a system of two bodies. Tsoi (1961 later considered transient heat transfer in a system of bodies. In an earlier paper the present authors (In course of publication) made an attempt to find the expressions for temperature fields in a system consisting of a finite rod and a semi-infinite rod under the boundary conditions of the fourth kind.

Here, by making use of a Laplace transform technique, we have obtained expressions for temperature fields in a system consisting of two semi-infinite rods, when the lateral surfaces of the rods are thermally insulated. Results have been discussed by considering particular cases, namely when first rod is heated by a (*i*) continuous heat source (*ii*) a discrete moving heat source. For solution in the latter case the properties of Dirac delta function have been used. The most important thing is that, all the results are in dimensionless form.

Formulation of the Problem

A system consisting of two semi-infinite rods with different thermal properties is taken. The lateral surfaces of the rods are thermally insulated. The initial temperature of the first rod is $f_1'(r)$ and that of second is zero. The first rod is subjected to a volume heat source $W(r, t)$. The breadths and thicknesses of the rods are small in comparison to their lengths . In this case the temperature drops over the thicknesses and breadths of the rods may be taken to be zero. Thus, the differential equations of heat conduction for the rods are :

$$\frac{\partial T_1(r, t)}{\partial t} = a_1 \frac{\partial^2 T_1(r, t)}{\partial r^2} + \frac{W(r, t)}{c\gamma}, \quad (t > 0,\ r > 0), \qquad \dots (1)$$

and

$$\frac{\partial T_2(r, t)}{\partial t} = a_2 \frac{\partial^2 T_2(r, t)}{\partial r^2}, \quad (t > 0, r > 0). \qquad \dots (2)$$

Since the rods are brought into contact, the boundary conditions can be taken as

$$T_1(0,t) = T_2(-0,t)$$

$$T_{1,r}(0,t) = -\frac{1}{K_\lambda}\, T_{2,r}(0,t), \qquad \dots (3)$$

$$T_{1,r}(+\infty,t) = T_{2,r}(-\infty,t) = 0.$$

The statement of the problem is completed by specifying the initial conditions

$$T_1(r,0) = f_1'(r)$$

and

$$T_2(r,0) = 0. \qquad \dots (4)$$

On introducing the non-dimensional variables defined in the nomenclature list we can write

$$\frac{\partial\theta_1(x,F_0)}{\partial F_0} = M_1\frac{\partial^2\theta_1(x,F_0)}{\partial x^2} + P_0(x,F_0), \qquad \dots (5)$$

$$\frac{\partial\theta_2(x,F_0)}{\partial F_0} = M_2\frac{\partial^2\theta_2(x,F_0)}{\partial x^2}, \qquad \dots (6)$$

subject to the initial conditions

$$\theta_1(x,0) = f_1(x)$$

$$\theta_2(x,0) = 0 \qquad \dots (7)$$

and boundary conditions

$$\theta_1(0,F_0) = \theta_2(-0,F_0),$$

$$\theta_{1,x}(\infty,F_0) = \theta_{2,x}(-\infty,F_0) = 0, \qquad \dots (8)$$

$$\partial_{1,x}(0,F_0) = -\frac{1}{K_\lambda}\,\theta_{2,x}(0,F_0).$$

The lower subscript x stands for the derivative with respect to x. The origin of the coordinate is considered at the point of contact of the rods.

Solution by Operational Method

Under the Laplace transform which is defined as

$$\phi(x,s) = \int_0^\infty \exp(-sF_0)\theta(x,F_0)\mathrm{d}F_0, \qquad \dots (9)$$

with the inversion

$$\theta(x,F_0) = \frac{1}{2\pi i'}\int_{\gamma-i'\infty}^{\gamma+i'\infty} \exp(sF_0)\phi(x,s)\mathrm{d}s, \qquad \dots (10)$$

where the integration is performed along a line $s = \gamma$ in the complex plane, the solution of the differential equations (5) and (6) under the conditions given in (7) and (8) will be

$$\phi_1(x, s) = \frac{1}{1+K_\varepsilon} [K_\varepsilon(M_1/s)^{\frac{1}{2}}\xi_{,x}(0, s)-\xi(0, s)]\exp(-x(s/M_1)^{\frac{1}{2}}+\xi(x, s) \quad \ldots \quad (11)$$

and

$$\phi_2(x, s) = \frac{K}{1+K_\varepsilon} [(M_1/s)^{\frac{1}{2}}\xi_{,x}(0, s)+\xi(0, s)]\exp(-|x|(s/M_2)^{\frac{1}{2}}, \quad \ldots \quad (12)$$

where,

$$\xi(x, s) = \frac{1}{D^2-s/M_1}\{-f_1(x)-\psi(x, s)\},$$

$$\psi(x, s) = \int_0^\infty \exp(-sF_0)P_0(x, F_0)\mathrm{d}F_0.$$

On taking inverse Laplace transorm, we have the temperature distribution in the first rod as

$$\theta_1(x, F_0) = \frac{K_\varepsilon}{1+K_\varepsilon} (M_1/\pi)^{\frac{1}{2}} \int_0^{F_0} \frac{\bar{\xi}_{,x}(0, F_0-u)}{u^{\frac{1}{2}}} \exp(-x^2/4M_1u)\mathrm{d}u$$

$$-\frac{x}{2(1+K_\varepsilon)(\pi M_1)^{\frac{1}{2}}} \int_0^{F_0} \frac{\bar{\xi}(0, F_0-u}{u^{3/2}} \exp(-x^2/4M_1u)\mathrm{d}u+\bar{\xi}(x, F_0) \quad \ldots \quad (13)$$

and temperature distribution in the second rod as

$$\theta_2(x, F_0) = \frac{K_\varepsilon}{1+K_\varepsilon} (M_1/\pi)^{\frac{1}{2}} \int_0^{F_0} \frac{\bar{\xi}_{,x}(0, F_0-u)}{u^{\frac{1}{2}}} \exp(-x^2/4M_2u)du$$

$$+\frac{K_\varepsilon|x|}{2(1+K_\varepsilon)(\pi M_2)^{\frac{1}{2}}} \int_0^{F_0} \frac{\bar{\xi}(0, F_0-u)}{u^{3/2}} \exp(-x^2/4M_2u)\mathrm{d}u, \quad \ldots \quad (14)$$

where

$$\xi(x, s) = \int_0^\infty \exp(-sF_0)\bar{\xi}(x, F_0)\mathrm{d}F_0$$

$$\xi_{,x}(x, s) = \int_0^\infty \exp(-sF_0)\bar{\xi}_{,x}(x, F_0)\mathrm{d}F_0.$$

Analysis

We shall illustrate this problem by considering three particular cases :

Case (*i*) : We cosider the case in which the source of heating is a periodic function of the generalized time F_0, *viz*,.

$$P_0(x, F_0) = P_{0c}(1-\cos lF_0), \quad \ldots \quad (15)$$

where l is a parameter Further, we assume that the initial temperature of the first rod is an exponentially decreasing function of the space coordinate x

$$f_1(x) = f_0(1-\exp(-bx)), \quad \ldots \quad (16)$$

where b is a parameter and f_0 is a constant initial temperature at the end of the rod

Thus, on using equations (15) and (16) in the equations (11) and (12), we get

$$\phi_1(x, s) = \frac{K_\varepsilon f_0}{2(1+K_\varepsilon)}\left(\frac{1}{s^{\frac{1}{2}}-bM_1^{\frac{1}{2}}}-\frac{1}{s^{\frac{1}{2}}+bM_1^{\frac{1}{2}}}\right)\frac{1}{s^{\frac{1}{2}}}\exp(-x(s/M_1)^{\frac{1}{2}})$$

$$+\frac{f_0}{(1+K_\varepsilon)}\cdot\frac{\exp(-x(s/M_1)^{\frac{1}{2}})}{(s-b^2M_1)}+\frac{P_{0c}}{1+K_\varepsilon}\cdot\frac{\exp(-x(s/M_1)^{\frac{1}{2}})}{s^2+l^2}$$

$$-\frac{f_0}{(1+K_\varepsilon)}\cdot\frac{\exp(-x(s/M_1)^{\frac{1}{2}})}{s}-\frac{P_{0c}}{(1+K_\varepsilon)}\cdot\frac{\exp(-x(s/M_1)^{\frac{1}{2}})}{s^2}$$

$$+\frac{f_0\exp(-bx)}{b^2M_1-s}-\frac{P_{0c}}{s^2+l^2}+\frac{P_{0c}}{s^2}+\frac{f_0}{s} \quad \ldots \quad (17)$$

and

$$\phi_2(x, s) = \frac{K_\varepsilon f_0}{2(1+K_\varepsilon)}\left(\frac{1}{s^{\frac{1}{2}}-bM_1^{\frac{1}{2}}}-\frac{1}{s^{\frac{1}{2}}+bM_1^{\frac{1}{2}}}\right)\exp(-|x|(s/M_2)^{\frac{1}{2}})$$

$$+\frac{K_\varepsilon}{1+K_\varepsilon}\left(\frac{f_0}{s}+\frac{f_0}{b^2M_1-s}+\frac{P_{0c}}{s^2}-\frac{P_{0c}}{s^2+l^2}\right)\exp(-|x|(s/M_2)^{\frac{1}{2}}) \quad \ldots \quad (18)$$

On taking inverse Laplace transform, we obtain the expressions for temperature fields in the rods as

$$\theta_1(x, F_0) = \tfrac{1}{2}f_0\exp(b^2M_1F_0-bx)\,\mathrm{erfc}\left(\frac{x-2bM_1F_0}{2(M_1F_0)^{\frac{1}{2}}}\right)$$

$$+\frac{1-K_\varepsilon}{2(1+K_\varepsilon)}f_0\exp(b^2M_1F_0+bx)\mathrm{erfc}\left(\frac{x+2bM_1F_0}{2(M_1F_0)^{\frac{1}{2}}}\right)$$

$$-\frac{f_0}{1+K_\varepsilon}\,\mathrm{erfc}\left(\frac{x}{2(M_1F_0)^{\frac{1}{2}}}\right)-\frac{4P_{0c}F_0}{1+K_\varepsilon}\,i^2\,\mathrm{erfc}\left(\frac{x}{2(M_1F_0)^{\frac{1}{2}}}\right)$$

$$+\frac{P_{0c}}{2l(1+K_\varepsilon)}\Big\{\exp(-x(l/2M_1)^{\frac{1}{2}})\sin(lF_0-x(l/2M_1)^{\frac{1}{2}})$$

$$\times\mathrm{erfc}\left(\frac{x}{2(M_1F_0)^{\frac{1}{2}}}-(lF_0/2)^{\frac{1}{2}}\right)+\exp(x(l/2M_1)^{\frac{1}{2}})\sin(lF_0+x(l/2M_1)^{\frac{1}{2}})$$

$$\times \operatorname{erfc}\left(\frac{x}{2(M_1 F_0)^{\frac{1}{2}}}+(lF_0 2)^{\frac{1}{2}}\right)+\frac{4}{\pi^{\frac{1}{2}}} \exp\left[-\left(\frac{x^2}{4M_1 F_0}+\frac{lF_0}{2}\right)^2\right]$$

$$\left.\times \int_0^{(\frac{1}{2}lF_0)^{\frac{1}{2}}} \sin(lF_0-y(2lF_0)^{\frac{1}{2}}) \sin(x(l/2M_1)^{\frac{1}{2}}-xy/(M_1F_0)^{\frac{1}{2}})\exp y^2\, dy\right\}$$

$$+P_{0c}(F_0-\frac{1}{l}\sin lF_0)+f_0(1-\exp(b^2M_1F_0-bx)) \quad \ldots \quad (19)$$

and

$$\theta_2(x, F_0) = -\frac{K_\varepsilon}{1+K_\varepsilon} f_0 \exp(b^2M_1F_0+b\,|x|\,(M_1/M_2)^{\frac{1}{2}})\operatorname{erfc}\left(\frac{|x|}{2(M_2F_0)^{\frac{1}{2}}}\right.$$

$$\left.+b(M_1F_0)^{\frac{1}{2}}\right)+\frac{K_\varepsilon}{1+K_\varepsilon} f_0 \operatorname{erfc}\frac{|x|}{2(M_2F_0)^{\frac{1}{2}}}$$

$$+\frac{4K_\varepsilon P_{0c}F_0}{1+K_\varepsilon} i^2 \operatorname{erfc}\left(\frac{|x|}{2(M_2F_0)^{\frac{1}{2}}}\right)-\frac{K_\varepsilon P_{0c}}{2l(1+K_\varepsilon)}\Big[\exp(-|x|\,l/2M_2)^{\frac{1}{2}}$$

$$\times \sin(lF_0-|x|\,(l/2M_2)^{\frac{1}{2}} \operatorname{erfc}\left(\frac{x}{2(M_2F_0)^{\frac{1}{2}}}-(\tfrac{1}{2}lF_0)^{\frac{1}{2}}\right)$$

$$+\exp(|x|\,(l/2M_2)^{\frac{1}{2}}) \sin(lF_0+|x|\,(l/2M_2)^{\frac{1}{2}})\operatorname{erfc}\left(\frac{|x|}{2(M_2F_0)^{\frac{1}{2}}}+(\tfrac{1}{2}lF_0)^{\frac{1}{2}}\right)$$

$$+\frac{4}{\pi^{\frac{1}{2}}} \exp\left(-\frac{x^2}{4M_2F_0}+\tfrac{1}{2}lF_0\right)\int_0^{(\frac{1}{2}lF)^{\frac{1}{2}}} \sin(lF_0-2(lF_0)^{\frac{1}{2}}\, y)$$

$$\times\sin(|x|\,(l/2M_2)^{\frac{1}{2}}-|x|\,y/(M_2F_0)^{\frac{1}{2}})\exp(y^2)dy\Big]. \quad \ldots \quad (20)$$

Solution for Small Values of Generalized Time

We know from the theory of Laplace transform

$$\underset{F_0\to 0}{\mathrm{Lt}}\ \theta(x, F_0) = \underset{s\to \infty}{\mathrm{Lt}}\ \bar{\phi}(x, s).$$

Using this for $\phi_1(x, s)$ and $\phi_2(x, s)$ and takingi nverse Laplace transform, we obtain the expressions of temperature fields for small values of the generalized time as

$$\theta_1(x, F_0) = \frac{2bf_0(M_1F_0)^{\frac{1}{2}}}{1+K_\varepsilon}\left\{K_\varepsilon\, i \operatorname{erfc}\left(\frac{x}{2(M_1F_0)^{\frac{1}{2}}}\right)+2b(M_1F_0)^{\frac{1}{2}} i^2 \operatorname{erfc}\left(\frac{x}{2(M_1F_0)^{\frac{1}{2}}}\right)\right\} \quad \ldots \quad (21)$$

and

$$\theta_2(x, F_0) = \frac{2bf_0K_\epsilon(M_1F_0)^{\frac{1}{2}}}{1+K_\epsilon}\left\{i \operatorname{erfc}\left(\frac{|x|}{2(M_2F_0)^{\frac{1}{2}}}\right)-2b(M_1F_0)^{\frac{1}{2}}\right.$$

$$\left.\times i^2 \operatorname{erfc}\left(\frac{|x|}{2(M_2F_0)^{\frac{1}{2}}}\right)\right\} \qquad (22)$$

Discussion

From the equations (19) and (20) with $x = 0$, we have the temperature fields at the point of contact of the rods as

$$\theta(0, F_0) = \theta_1(0, F_0) = \theta_2(0, F_0)$$

$$= \frac{K_\epsilon}{1+K_\epsilon}\left[f_0\{1-\exp(b^2M_1F_0)\operatorname{erfc}(b(M_1F_0)^{\frac{1}{2}}\}+P_{o_c}\left\{F_0-\frac{\sin lF_0}{l}\right\}\right] \quad \ldots \quad (23)$$

For example, if the rods are of copper and cast iron, we have $a_1 = 1.14$, $a_2 = 0.12$, $K_\epsilon = 2.51$ Further, if we assume $a = l = 1$ and $P_{oc} = 2f_0$, we have on substitution the expression for the temperature fields at the point of contact

$$\frac{\theta(0, F_0)}{f_0} = \frac{2.51}{3.51}\left[1-\exp(1.14b^2F_0)\operatorname{erfc}(b(1.14F_0)^{\frac{1}{2}}+2F_0-2\sin F_0\right].$$

A graph is plotted between $\dfrac{\theta(0, F_0)}{f_0}$ and F_0 for different values of b as shown in figure 1. From this we observe that for finite values of b at a fixed value of F_0, the slopes of the curves increase as b increases and at $b = \infty$ the corresponding curve becomes parallel to the F_0 axis.

Case (ii) : Here we consider that the first rod is heated by a discrete moving heat source

$$P_0(x, F_0) = \sum_{j=1}^{n} P_{0cj}\delta(x-x_j)\delta(F_0-F_{0j}) \qquad \ldots \quad (25)$$

and the initial temperature of the first rod is f_0 :

$$f_1(x) = f_0.$$

Thus, on using equations (25) and (26) in equations (11) and (12), we get

$$\phi_1(x, s) = \frac{1}{2(M_1)^{\frac{1}{2}}}\sum_{j=1}^{n}\frac{P_{0cj}}{s^{\frac{1}{2}}}\{\exp(-(x-x_j)(s/M_1)^{\frac{1}{2}}-sF_{0j})$$

$$-\exp(-(x+xj)(s/M_1)^{\frac{1}{2}}-sF_{0j})\}-\frac{f_0}{(1+K_\epsilon)}\,\frac{1}{s}\exp(-x(s/M_1)^{\frac{1}{2}})+\frac{f_0}{s} \qquad \ldots \quad (27)$$

and

$$\phi_2(x, s) = \frac{K_s}{1+K_r} \cdot \frac{f_0}{s} \exp(-|x|(s/M_2)^{\frac{1}{2}}) \quad \dots \quad (28)$$

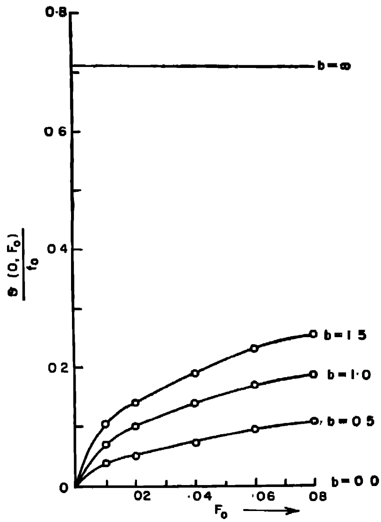

FIG 1 VARIATION OF $\frac{\theta(0,F_0)}{f_0}$ VERSUS F_0 AT DIFFERENT b.

On taking inversion of the equation (27) and (28) and using

$$\int_0^\infty \delta(t-a)\, G(t)\mathrm{d}t = G(a),$$

where $G(t)$ is any continuous function, we obtain

$$\theta_1(x, F_0) = \frac{1}{2(\pi M_1)^{\frac{1}{2}}} \sum_{j=1}^{n} \frac{P_{0cj}}{(F_0-F_{0j})^{\frac{1}{2}}} \left\{ \exp\left(-\frac{(x-x_j)^2}{4M_1(F_0-F_{0j})}\right) - \exp\left(-\frac{(x+x_j)^2}{4M_1(F_0-F_{j0})}\right)\right\} + f_0 \left\{1-\frac{1}{1+K_s} \operatorname{erfc}\left(\frac{x}{2(M_1F_0)^{\frac{1}{2}}}\right)\right\} \quad \dots \quad (29)$$

and

$$\theta_2(x, F_0) = \frac{K_s}{1+K_s} f_0 \operatorname{erfc}\left(\frac{|x|}{2(M_2F_0)^{\frac{1}{2}}}\right). \quad \dots \quad (30)$$

If we consider that the source in the first rod is acting at the points $x = 1$ and $x = 2$ at $F_0 = 0.01$ and $F_0 = 0.02$ respectively, we have the expressions for temperature fields for the rods consisting of copper and cast iron as

$$\frac{\theta_1(x, F_0)}{f_0} = 0.53 \left\{ \frac{1}{(F_0 - 0.01)^{\frac{1}{2}}} \left(\exp\left(- \frac{(x-1)^2}{4.56(F_0 - 0.01)} \right) - \exp\left(- \frac{(x+1)^2}{4.56(F_0 - 0.01)} \right) \right) + \frac{1}{(F_0 - 0.02)^{\frac{1}{2}}} \left(\exp\left(\frac{-(x-2)^2}{4.56(F_0 - 0.02)} \right) - \exp\left(- \frac{(x+2)^2}{4.56(F_0 - 0.02)} \right) \right) \right\} - \frac{1}{3.51} \operatorname{erfc}\left(\frac{x}{2(1.14F_0)^{\frac{1}{2}}} \right) + 1 \quad \ldots \quad (31)$$

and

$$\frac{\theta_2(x, F_0)}{f_0} = \frac{2.51}{3.51} \operatorname{erfc}\left(\frac{|x|}{2(0.12F_0)^{\frac{1}{2}}} \right). \quad (32)$$

A graph is plotted between $\frac{\theta_i(x, F_0)}{f_0}$ ($i = 1$ for $x > 0$, $i = 2$ for $x > 0$) and x for various values of the generalized time in the range $0 \leqslant F_0 \leqslant \infty$ as shown in figure 2. From this, we observed that for finite values of generalized time the temperature of the first rod is always greater than that of temperature of the point of contact while that of second is less and at $F_0 = \infty$, the temperature of both rods becomes equal. Further, we see that the temperature of the point of contact is constant during the whole heat transfer process.

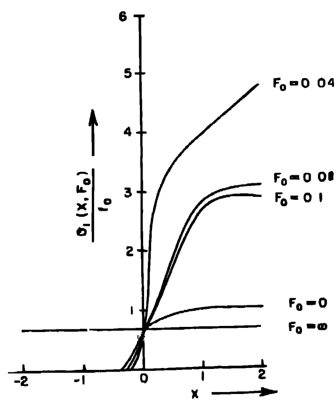

FIG 2 VARIATION OF $\frac{\theta_1(X, F_0)}{f_0}$ VERSUS X AT DIFFERENT F_0

Such type of the problems occurs in many practical cases such as frictional heating, grinding, machining, surface heating of a moving object, flame cutting and welding etc.

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